

Nash Equilibria in Games with Playerwise Concave Coupling Constraints: Existence and Computation

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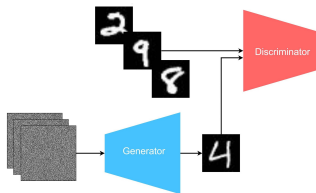
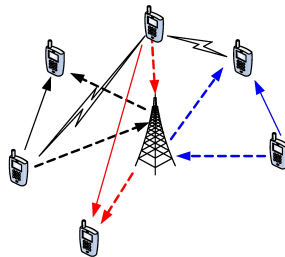
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Games with coupling constraints

① existence and ② computation of Nash equilibria

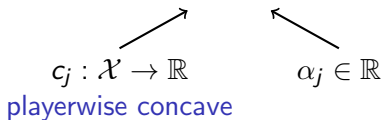


Game setting

- m players, $\mathcal{X}_i \subset \mathbb{R}^{d_i}$ convex compact $\forall i \in [m]$
- joint strategy space $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$
- utilities $u_i : \mathcal{X} \rightarrow \mathbb{R}$ for $i = 1, \dots, m$
 - ▶ **playerwise concave:** $u_i(\cdot, x_{-i})$ concave $\forall x_{-i} \in \mathcal{X}_{-i}, \forall i \in [m]$

Coupling constraints:

- joint feasible set $\mathcal{C} := \{x \in \mathcal{X} \mid c_j(x) \geq \alpha_j, \forall j \in [b]\}$

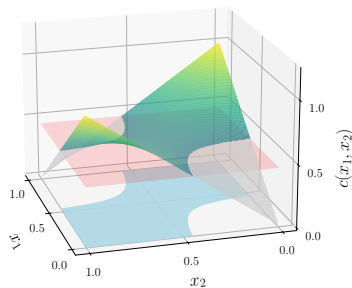


- $x^* \in \mathcal{C}$ is **constrained Nash equilibrium** if $\forall i \in [m], \forall x'_i \in \mathcal{X}_i$ with $(x'_i, x_{-i}^*) \in \mathcal{C}$,

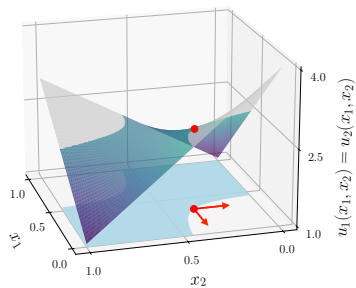
$$u_i(x'_i, x_{-i}^*) \leq u_i(x^*)$$

Illustrative example

- $\mathcal{X}_1 = \mathcal{X}_2 = [0, 1]$
- $u_1(x_1, x_2) = u_2(x_1, x_2) = \begin{bmatrix} x_1 & 1 - x_1 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_2 \\ 1 - x_2 \end{bmatrix}$
- $c(x_1, x_2) = \begin{bmatrix} x_1 & 1 - x_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ 1 - x_2 \end{bmatrix}, \quad \alpha = \frac{1}{2}$



feasible region $\mathcal{C}$



utilities (constrained to $\mathcal{C}$)

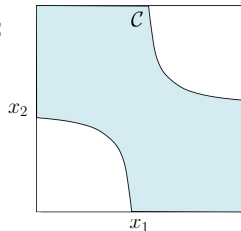
① **Existence** of constrained Nash equilibria

② **Computation** of constrained Nash equilibria

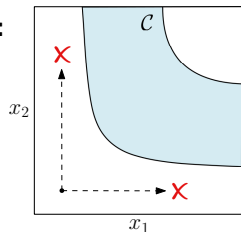
Prior results

- [Ros65]: existence for playerwise concave utilities and **convex** feasible region $\mathcal{C}$
- [Deb52]: existence under a **feasible response** assumption:
 - ▶ $\forall x \in \mathcal{X}, \forall i \in [m], \exists x'_i$ such that $(x'_i, x_{-i}) \in \mathcal{C}$

violated:



violated:



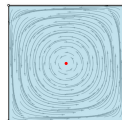
Our result

Theorem

Let *utilities and constraints* be continuous and *playerwise concave* and $\mathcal{C} \neq \emptyset$. Then, under a *playerwise MFCQ* assumption, there exists a constrained Nash equilibrium $x^* \in \mathcal{C}$.

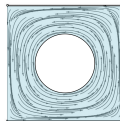
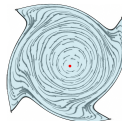
- Without coupling constraints:

- ▶ define $\Psi : \mathcal{X} \rightarrow \mathcal{X}$ with
 $\Psi(x)_i = \text{"best response of player } i \text{ to } x_{-i}"$
- ▶ Ψ has fixed point by [Kak41]'s theorem (as $\mathcal{X}$ is **convex**)



- With coupling constraints:

- ▶ define $\Psi : \mathcal{C} \rightarrow \mathcal{C}$



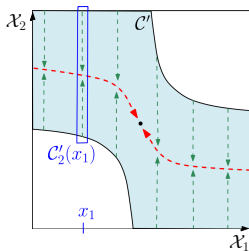
Does $\mathcal{C}$ have sufficient structure for existence of fixed point?

Our result

Theorem

Let *utilities and constraints* be continuous and *playerwise concave* and $C \neq \emptyset$. Then, under a *playerwise MFCQ* assumption, there exists a constrained Nash equilibrium $x^* \in C$.

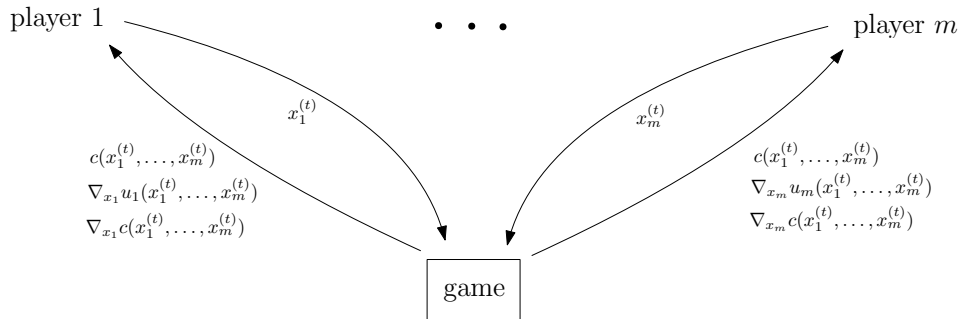
- [Beg50]: continuous function over compact contractible (no holes) domain has fixed point
- **key lemma:** playerwise concave constraints $\implies$ contractibility of each connected component C' of C



① **Existence** of constrained Nash equilibria

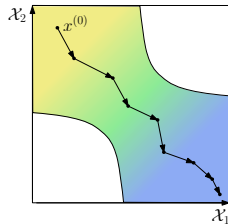
② **Computation** of constrained Nash equilibria

Independent learning protocol



Interior point method

Starting from feasible initial strategy $x^{(0)} \in \mathcal{C}$:



Log barrier regularization:

$$B_i^\eta(x) := u_i(x) + \eta \log(c(x) - \alpha), \quad i \in [m], \eta > 0$$

Players simultaneously update

$$x_i^{(t+1)} = \mathcal{P}_{\mathcal{X}_i} \left[x^{(t)} + \gamma^{(t)} \nabla_{x_i} B_i^\eta(x^{(t)}) \right]$$

for stepsize $\gamma^{(t)} > 0$.

Convergence result

- Nash equilibrium computation is PPAD-hard (even in normal-form games [DGP09])

Assumption

There exists a function $\Phi : \mathcal{C} \rightarrow \mathbb{R}$ such that for all $i \in [m]$, $x \in \mathcal{C}$, and $x'_i \in \mathcal{C}_i(x_{-i})$,

$$\Phi(x) - \Phi(x'_i, x_{-i}) = u_i(x) - u_i(x'_i, x_{-i}).$$

- $\Phi^\eta(x) := \Phi(x) + \eta \log(c(x) - \alpha)$ is a potential function for the regularized game
- ϵ -stationary point of Φ^η is constrained ϵ -Nash equilibrium

Theorem (informal)

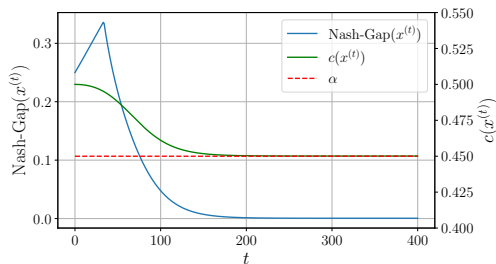
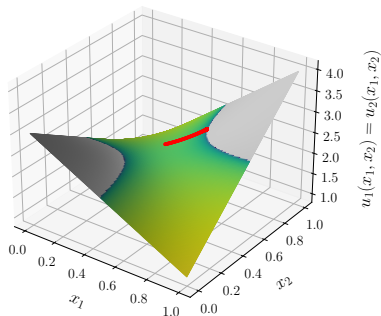
Let $\epsilon > 0$. Suppose

- c and u_i *playerwise concave, smooth, and Lipschitz* for all $i \in [m]$,
- regularization $\eta = \epsilon$; suitable stepsizes $\{\gamma^{(t)}\}_{t=0}^{T-1}$.

After $T = \mathcal{O}(\epsilon^{-3})$ iterations of independent log barrier regularized gradient ascent,
 $\exists t \in \{0, \dots, T\}$ such that $x^{(t)}$ is a *constrained ϵ -approximate Nash equilibrium*.

Simulation

- $\mathcal{X}_1 = \mathcal{X}_2 = [0, 1]$
- $u_1(x_1, x_2) = u_2(x_1, x_2) = \begin{bmatrix} x_1 & 1 - x_1 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_2 \\ 1 - x_2 \end{bmatrix}$
- $c(x_1, x_2) = \begin{bmatrix} x_1 & 1 - x_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ 1 - x_2 \end{bmatrix}, \quad \alpha = \frac{1}{2}$



Conclusion

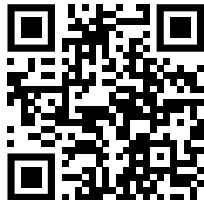
Summary:

- **existence** of Nash equilibria for games with playerwise concave utilities and constraints
- under potential structure, playerwise independent **algorithm** with finite time convergence

Outlook:

- existence of Nash equilibria for constrained Markov games (no playerwise concavity)
- computation beyond potential games (e.g., zero-sum)
- learning setting (without exact gradient feedback)

Poster #4507



References I



Edward G Begle.

A fixed point theorem.

Annals of Mathematics, 51(3):544–550, 1950.



Gerard Debreu.

A social equilibrium existence theorem.

Proceedings of the national academy of sciences, 38(10):886–893, 1952.



Constantinos Daskalakis, Paul W Goldberg, and Christos H Papadimitriou.

The complexity of computing a Nash equilibrium.

Communications of the ACM, 52(2):89–97, 2009.



Shizuo Kakutani.

A generalization of brouwer's fixed point theorem.

1941.

References II



J Ben Rosen.

Existence and uniqueness of equilibrium points for concave n -person games.

Econometrica: Journal of the Econometric Society, pages 520–534, 1965.



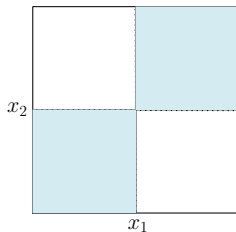
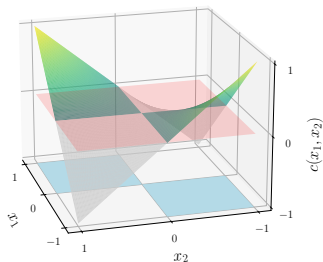
Arda Sahiner, Tolga Ergen, Batu Ozturkler, Burak Bartan, John M Pauly, Morteza Mardani, and Mert Pilanci.

Hidden convexity of Wasserstein GANs: Interpretable generative models with closed-form solutions.

In *International Conference on Learning Representations*, 2022.

Appendix

Playerwise MFCQ assumption



Assumption

Let $\mathcal{J}(x) := \{j \in [b] \mid c_j(x) = \alpha_j\}$ be the set of active constraints for a feasible strategy $x \in \mathcal{C}$. We assume that for all $x \in \mathcal{C}$ and $i \in [m]$, there exists $\delta_i \in \mathbb{R}^{d_i}$ with $x_i + \delta_i \in \mathcal{X}_i$ and $\|\delta_i\| \leq 1$, such that for all $j \in \mathcal{J}(x)$, we have $\langle \delta_i, \nabla_{x_i} c_j(x) \rangle > 0$.